Binary operations

Algebra is essentially the study of sets equipped with various "binary operations."

Examples:

The following are familiar binary operations:

1.) $+, \cdot, -$ on $\mathbb{R}$ (or $\mathbb{Q}$ or $\mathbb{Z}$)

2.) $\div$ on $\mathbb{R} - \{0\}$ or $\mathbb{Q} - \{0\}$ (why not $\mathbb{Z} - \{0\}$?)

3.) Addition "mod 3". i.e. $\{0, 1, 2\}$, where $a + b$ = remainder when dividing by 3.

So the addition table looks like

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

4.) min is a binary operation on $\mathbb{R}$ defined $\min(a,b) = \begin{cases} a & \text{if } a \le b \\ b & \text{if } b \le a \end{cases}$

We can also come up w/ binary operations:

5.) Define $*$ on $\mathbb{R}$ to be $a * b = 2a + 5b$.

Def: If A is a set, then a <u>binary operation</u> $*$ on A is a function

$$* : A \times A \longrightarrow A$$

Denote $*(a,b)$ by $a * b$.

Just like the binary operations with which we're already familiar, we mostly care about ones that satisfy nice properties.

Def: $*$ on A is <u>commutative</u> if $a*b = b*a \quad \forall\, a,b \in A$.

Ex: Let $A = \{ f : \mathbb{R} \to \mathbb{R} \}$ = the set of functions from $\mathbb{R}$ to $\mathbb{R}$.

$+$, $\cdot$, and $\circ$ are all binary operations on A.

$f+g$ is defined $(f+g)(x) = f(x) + g(x)$

$f \cdot g$ is defined $(f \cdot g)(x) = f(x) \cdot g(x)$

$f \circ g$ is defined $(f \circ g)(x) = f(g(x))$

In this case, $+$ and $\cdot$ are both commutative, but if

$$f(x) = x^2 \text{ and } g(x) = x+1, \text{ then}$$

$(f \circ g)(x) = (x+1)^2$ and $(g \circ f)(x) = x^2 + 1$, so $f \circ g \neq g \circ f$,

so $\circ$ is not commutative.

Def: $*$ is <u>associative</u> if $\forall\, a,b,c \in A$, $(a*b)*c = a*(b*c)$.

Ex: $\cdot$ on $\mathbb{R}$ is associative: $(ab)c = a(bc)$. However, $\div$ on $\mathbb{R} - \{0\}$ is not associative: $(2/1)/2 = 2/2 = 1$ but $2/(1/2) = 4$.